

# Harmonic superspace, minimal unitary representations and quasiconformal groups

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**ABSTRACT:** We show that there is a remarkable connection between the harmonic superspace (HSS) formulation of  $N = 2$ ,  $d = 4$  supersymmetric quaternionic Kähler sigma models that couple to  $N = 2$  supergravity and the minimal unitary representations of their isometry groups. In particular, for  $N = 2$  sigma models with quaternionic symmetric target spaces of the form  $G/H \times SU(2)$  we establish a one-to-one mapping between the Killing potentials that generate the isometry group  $G$  under Poisson brackets in the HSS formulation and the generators of the minimal unitary representation of  $G$  obtained by quantization of its geometric realization as a quasiconformal group. Quasiconformal extensions of U-duality groups of four dimensional  $N = 2$ ,  $d = 4$  Maxwell-Einstein supergravity theories (MESGT) had been proposed as spectrum generating symmetry groups earlier. We discuss some of the implications of our results, in particular, for the BPS black hole spectra of 4d  $N = 2$  MESGTs.

**KEYWORDS:** Black Holes in String Theory, Extended Supersymmetry, Supersymmetry and Duality, Supergravity Models.

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## 1. Introduction

The target manifolds of  $N = 2$  supersymmetric  $\sigma$  models coupled to supergravity in  $d = 4$  were shown to be quaternionic Kähler manifolds long time ago [1]. Later the results of [1] were reformulated in harmonic superspace [2, 3] as well as in projective superspace [4]. Some of these theories arise as subsectors of the low energy effective theories of type IIA or type IIB superstring compactified over a Calabi-Yau threefold. More specifically, type IIA (IIB) theories over Calabi-Yau manifolds lead to  $d = 4$ ,  $N = 2$  supergravity coupled to  $h_{(1,1)}[h_{(2,1)}]$  vector multiplets and  $(h_{(2,1)} + 1)[(h_{(1,1)} + 1)]$  hypermultiplets. Under further dimensional reduction to three dimensions the vector moduli spaces can be written as quaternionic Kähler manifolds (c-map) [5, 6] and hence the moduli spaces become products of two quaternionic Kähler manifolds in  $d = 3$ .

In this paper we study the symmetries of the  $N = 2$  supersymmetric  $\sigma$  models with quaternionic Kähler target manifolds that couple to supergravity in  $d = 4$  using the harmonic superspace (HSS) approach and show that there is a remarkable mapping between the realization of the symmetries in the HSS formulation and the minimal unitary realizations of their isometry groups. This mapping is made readily manifest within the formulation of minimal unitary realizations of noncompact simple groups obtained by quantization of their geometric realizations as quasiconformal groups [7–11]. For  $N = 2$  supersymmetric  $\sigma$  models the relevant real forms are quaternionic.

The plan of the paper is as follows. Section 2 is devoted to a review of the formulation of  $N = 2$  supersymmetric quaternionic Kähler  $\sigma$  models in harmonic superspace and implementation of their isometry groups through Killing potentials. In section 3 we review

the quasiconformal realizations of noncompact groups that can be formulated simply and in full generality using the Freudenthal triple systems associated with them. The quasiconformal groups can all be defined as invariance groups of generalized lightcones defined with respect to a quartic norm. In section 4 we review the construction of the minimal unitary representations of noncompact groups by quantization of their geometric realizations as quasiconformal groups. In section 5 we show that for a quaternionic symmetric  $N = 2$   $d = 4$   $\sigma$  model there is a precise mapping between the Killing potentials of its isometry group  $G$  and the generators of the minimal unitary realization of  $G$  given in sections 2 and 4, respectively. In section 6 we discuss some of the implications of our results and open problems. In particular, we discuss the implications for the proposal that the quasiconformal extensions of U-duality groups of  $d = 4$  Maxwell-Einstein supergravity theories act as spectrum generating symmetry groups.

## 2. $N = 2$ , $d = 4$ supersymmetric $\sigma$ -models in harmonic superspace

The target spaces of  $N = 2$  supersymmetric  $\sigma$ -models coupled to  $N = 2$  supergravity are quaternionic Kähler manifolds [1]. In this section we shall review the formulation of these  $\sigma$ -models in  $N = 2$ ,  $d = 4$ , harmonic superspace [12–14] following closely [15]. In the harmonic superspace approach the metric on a quaternionic target space is determined by a quaternionic potential  $\mathcal{L}^{+4}$ , which plays the same role for quaternionic Kähler manifolds as the Kähler potential does for Kähler manifolds.

The  $N = 2$  harmonic superspace action for the general  $4n$ -dimensional quaternionic  $\sigma$ -model is given by [15]

$$S = \int d\zeta^{-4} du \{ Q_{\alpha}^{+} D^{++} Q^{+\alpha} - q_i^{+} D^{++} q^{+i} + \mathcal{L}^{+4}(Q^{+}, q^{+}, u^{-}) \}. \quad (2.1)$$

where the integration is over the analytic superspace coordinates  $\zeta, u_i^{\pm}$ . The  $Q_{\alpha}^{+}(\zeta, u)$ ,  $\alpha = 1, \dots, 2n$  and the supergravity hypermultiplet compensators  $q_i^{+}(\zeta, u)$ , ( $i = 1, 2$ ) are analytic  $N = 2$  superfields. The  $u_i^{\pm}$ , ( $i = 1, 2$ ) are the  $S^2 = \frac{SU(2)_A}{U(1)}$  isospinor harmonics that satisfy

$$u^{+i} u_i^{-} = 1$$

and  $D^{++}$  is a supercovariant derivative with respect to harmonics with the property

$$D^{++} u_i^{-} = u_i^{+}$$

We recall that the analytic subspace of  $N = 2$  harmonic superspace involves only half the Grassmann variables with coordinates  $\zeta^M$  and  $u_i^{\pm}$

$$\zeta^M := \{ x_A^{\mu}, \theta^{a+}, \bar{\theta}^{\dot{a}+} \} \quad (2.2)$$

where

$$\begin{aligned}
 x_A^\mu &:= x^\mu - 2i\theta^{(i}\sigma^\mu\bar{\theta}^{j)}u_i^+u_j^- \\
 \theta^{a+} &:= \theta^{ai}u_i^+ \\
 \bar{\theta}^{\dot{a}+} &:= \bar{\theta}^{\dot{a}i}u_i^+ \\
 \theta^{(i}\sigma^\mu\bar{\theta}^{j)}u_i^+u_j^- &:= \theta^{(ai}(\sigma^\mu)_{a\dot{a}}\bar{\theta}^{\dot{a}j)}u_i^+u_j^- \\
 \mu &= 0, 1, 2, 3; \quad a = 1, 2; \quad \dot{a} = 1, 2
 \end{aligned}$$

This analytic subspace does not involve U(1) charge  $-1$  projections of the Grassmann variables and is closed under  $N = 2$  supersymmetry transformations. Furthermore it is "real" with respect to the conjugation  $\sim$

$$\begin{aligned}
 \tilde{x}^\mu &= x^\mu \\
 \tilde{\theta}^+ &= \bar{\theta}^+ \\
 \tilde{\bar{\theta}}^+ &= -\theta^+ \\
 \tilde{u}^{i\pm} &= u_i^\pm \\
 \tilde{u}_i^\pm &= -u^{i\pm}
 \end{aligned} \tag{2.3}$$

which is a product of complex conjugation and anti-podal map on the sphere  $S^2$ . For a complete description of harmonic superspace we refer to the monograph [2].

The quaternionic potential  $\mathcal{L}^{+4}$  is a homogeneous function in  $Q^+$  and  $q^+$  of degree two and has U(1)-charge  $+4$ . It does not depend on  $u^+$  and is an arbitrary "real function" otherwise, with the reality being defined with respect to the involution  $\sim$ . For simplicity we shall suppress the dependence of all the fields on the harmonic superspace coordinates  $\zeta^M$  and  $u_i^\pm$ .

As was first pointed out in [16] and later elaborated in [15, 2] the action (2.1) has a remarkable analogy to the Hamiltonian mechanics with the harmonic derivative  $D^{++}$  playing the role of time derivative. The superfields  $Q^+$  and  $q^+$  correspond to phase space coordinates and the Poisson brackets are given by

$$\{f, g\}^{--} = \frac{1}{2}\Omega^{\alpha\beta} \frac{\partial f}{\partial Q^{+\alpha}} \frac{\partial g}{\partial Q^{+\beta}} - \frac{1}{2}\epsilon^{ij} \frac{\partial f}{\partial q^{+i}} \frac{\partial g}{\partial q^{+j}}, \tag{2.4}$$

where  $\Omega^{\alpha\beta}$  and  $\epsilon^{ij}$  are the invariant antisymmetric tensors of  $Sp(2n)$  and  $Sp(2)$ , respectively. The indices are raised and lowered by these tensors

$$\begin{aligned}
 Q^{+\alpha} &= \Omega^{\alpha\beta} Q_\beta^+ \\
 q^{+i} &= \epsilon^{ij} q_j^+
 \end{aligned} \tag{2.5}$$

which satisfy<sup>1</sup>

$$\Omega^{\alpha\beta}\Omega_{\beta\gamma} = \delta_\gamma^\alpha \tag{2.6}$$

$$\epsilon^{ij}\epsilon_{jk} = \delta_k^i \tag{2.7}$$

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<sup>1</sup>Note that the conventions of [15] which we follow in this section differ from those of [11].

Because of this analogy and following [15] we shall refer to the quaternionic potential  $\mathcal{L}^{+4}$  as the Hamiltonian.

Isometries of the  $\sigma$ -model (2.1) are generated by Killing potentials  $K_A^{++}(Q^+, q^+, u^-)$  that obey the conservation law

$$\partial^{++} K_A^{++} + \{K_A^{++}, \mathcal{L}^{+4}\}^{--} = 0 \quad (2.8)$$

where  $\partial_{++}$  is defined as

$$\partial_{++} = u^{+i} \frac{\partial}{\partial u^{-i}}$$

The Killing potentials form the Lie algebra of the isometry group

$$\{K_A^{++}, K_B^{++}\}^{--} = f_{AB}{}^C K_C^{++} . \quad (2.9)$$

under the Poisson brackets (2.4).

The "Hamiltonians" of  $N = 2$   $\sigma$ -models coupled to  $N = 2$  supergravity with symmetric target manifolds were given in [15]. The quaternionic symmetric spaces, sometimes known as Wolf spaces [17], that are relevant to supergravity are of the non-compact type. For each simple Lie group there is a unique non-compact quaternionic symmetric space. A complete list of these spaces is given below.

$$\frac{\frac{SU(n, 2)}{U(n) \times Sp(2)}}{\frac{F_{4(+4)}}{Sp(6) \times Sp(2)}} \quad \frac{\frac{SO(n, 4)}{SO(n) \times SU(2) \times Sp(2)}}{\frac{E_{6(+2)}}{SU(6) \times Sp(2)}} \quad \frac{\frac{USp(2n, 2)}{Sp(2n) \times Sp(2)}}{\frac{E_{7(-5)}}{SO(12) \times Sp(2)}} \quad \frac{\frac{G_{2(+2)}}{SU(2) \times Sp(2)}}{\frac{E_{8(-24)}}{E_7 \times Sp(2)}} \quad (2.10)$$

which are all of the form  $G/H \times Sp(2)$  with  $H \subset Sp(2n)$ .

For a given quaternionic symmetric target space  $G/H \times Sp(2)$  of  $N = 2$   $\sigma$  model coordinatized by  $Q^+$  and  $q^+$ , every generator  $\Gamma_A$  of  $G$  maps to a function  $K_A^{++}(Q^+, q^+, u^-)$  such that the action of  $K_A^{++}$  is given via the Poisson brackets (2.4). The authors of [15] showed that the Hamiltonian  $\mathcal{L}^{+4}$  depends only on  $Q^+$  and the combination  $q^+ u^- := q^{+i} u_i^-$ ,

$$\mathcal{L}^{+4} = \mathcal{L}^{+4}(Q^+, (q^+ u^-)). \quad (2.11)$$

and can be written as

$$\mathcal{L}^{+4} = \frac{P^{+4}(Q^+)}{(q^+ u^-)^2} \quad (2.12)$$

The fourth order polynomial  $P^{+4}$  is given by

$$P^{+4}(Q^+) = \frac{1}{12} S_{\alpha\beta\gamma\delta} Q^{+\alpha} Q^{+\beta} Q^{+\gamma} Q^{+\delta} \quad (2.13)$$

where  $S_{\alpha\beta\gamma\delta}$  is a completely symmetric invariant tensor of  $H$ . In terms of matrices  $t_{\alpha\beta}^a$ ,  $a = 1, 2, \dots, \dim(H)$  representing the action of the Lie algebra  $\mathfrak{h}$  of  $H$  on  $Q^{+\alpha}$  the invariant tensor reads as [15]

$$S_{\alpha\beta\gamma\delta} = h_{ab} t_{\alpha\beta}^a t_{\gamma\delta}^b + \Omega_{\alpha\gamma} \Omega_{\beta\delta} + \Omega_{\alpha\delta} \Omega_{\beta\gamma} . \quad (2.14)$$

where  $h_{ab}$  is the Killing metric of  $H$ .

The Killing potentials that generate the isometry group  $G$  are [15]

$$\mathbf{Sp}(2) : K_{ij}^{++} = 2(q_i^+ q_j^+ - u_i^- u_j^- \mathcal{L}^{+4}), \tag{2.15}$$

$$\mathbf{H} : K_a^{++} = t_{\alpha\beta} Q^{+\alpha} Q^{+\beta}, \tag{2.16}$$

$$\mathbf{G}/\mathbf{H} \times \mathbf{Sp}(2) : K_{i\alpha}^{++} = 2q_i^+ Q_\alpha^+ - u_i^- (q^+ u^-) \partial_\alpha^- \mathcal{L}^{+4}, \tag{2.17}$$

where

$$\begin{aligned} \partial_\alpha^- &:= \frac{\partial}{\partial Q^{+\alpha}} \\ t_{\alpha\beta} &= \Omega_{\beta\gamma} t_{\alpha\gamma} \end{aligned}$$

The  $Sp(2)$  potentials  $K_{ij}^{++}$  are conserved for an arbitrary polynomial  $P^{+4}(Q^+)$ .  $t^a$  are the representation matrices of the generators of  $H$  acting on  $Q^{+\alpha}$ . This implies that the fourth order polynomial  $P^{+4}$  is proportional to the quadratic "Casimir function"  $h^{ab} K_a^{++} K_b^{++}$  of  $H$ . Furthermore,  $P^{+4}$  can also be expressed in terms of the coset Killing potentials, or the  $Sp(2)$  Killing potentials as follows [15] :

$$P^{+4} = -\frac{1}{16} \epsilon^{ij} \Omega^{\alpha\beta} K_{i\alpha}^{++} K_{j\beta}^{++} = -\frac{1}{8} K^{++ij} K_{ij}^{++}. \tag{2.18}$$

### 3. Freudenthal triple systems and quasiconformal group actions

Lie algebra  $\mathfrak{g}$  of every simple Lie group  $G$  admits a 5-graded decomposition of the form

$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2}. \tag{3.1}$$

such that grade  $\pm 2$  subspaces are one dimensional.<sup>2</sup> The grade zero algebra  $\mathfrak{g}^0$  has the form

$$\mathfrak{g}^0 = \mathfrak{h} \oplus \Delta \tag{3.2}$$

where  $\Delta$  is the generator that determines the 5-grading. The grade  $\pm 2$  generators and  $\Delta$  generate a distinguished  $sl(2)$  subalgebra of  $\mathfrak{g}$ . We shall denote the subgroup generated by  $\mathfrak{h}$  as  $H$ . A simple Lie algebra with such a 5-graded decomposition can always be constructed over a Freudenthal triple system  $\mathcal{F}$  [18, 19]. Freudenthal introduced these triple systems in his study of the metasymplectic geometries associated with exceptional groups [18]. A Freudenthal triple system (FTS) is a vector space  $\mathcal{F}$  with a trilinear product  $(X, Y, Z)$  and a skew symmetric bilinear form  $\langle X, Y \rangle$  that satisfy:<sup>3</sup>

$$\begin{aligned} (X, Y, Z) &= (Y, X, Z) + 2 \langle X, Y \rangle Z, \\ (X, Y, Z) &= (Z, Y, X) - 2 \langle X, Z \rangle Y, \\ \langle (X, Y, Z), W \rangle &= \langle (X, W, Z), Y \rangle - 2 \langle X, Z \rangle \langle Y, W \rangle, \\ (X, Y, (V, W, Z)) &= (V, W, (X, Y, Z)) + ((X, Y, V), W, Z) \\ &\quad + (V, (Y, X, W), Z). \end{aligned} \tag{3.3}$$

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<sup>2</sup>Of course for  $sl(2)$  this 5-grading degenerates to a 3-grading.

<sup>3</sup>It should be noted that the triple product can be modified by terms involving the symplectic invariant, such as  $\langle X, Y \rangle Z$ . The choice given above was made in [7].

A quartic invariant  $\mathcal{I}_4$  can be defined over the FTS  $\mathcal{F}$  using the triple product and the bilinear form as

$$\mathcal{I}_4(X) := \frac{1}{48} \langle (X, X, X), X \rangle \tag{3.4}$$

which is invariant under the automorphism group  $H = \text{Aut}(\mathcal{F})$  of  $\mathcal{F}$ .

In the corresponding construction of  $\mathfrak{g}$  over  $\mathcal{F}$ , the generators of grade  $\pm 1$  subspaces of  $\mathfrak{g}$  are labelled by the elements of  $\mathcal{F}$  and all the commutation relations are expressed in terms of the triple product  $(X, Y, Z)$  [19]. Following [7] let us denote the Lie algebra generators belonging to grade  $+1$  and grade  $-1$  subspaces as  $U_A$  and  $\tilde{U}_A$ , respectively, where  $A \in \mathcal{F}$ . The five grading now reads as

$$\mathfrak{g} = \tilde{K}_{AB} \oplus \tilde{U}_A \oplus S_{AB} \oplus U_A \oplus K_{AB}$$

where  $A, B \in \mathcal{F}$ . The symplectic trace of  $S_{AB}$  is the generator  $\Delta$  that determines the five grading [20]. Since they are one dimensional the grade  $\pm 2$  generators  $K_{AB}$  and  $\tilde{K}_{AB}$  labeled by two elements can be written as

$$K_{AB} = \langle A, B \rangle K \quad \tilde{K}_{AB} = \langle A, B \rangle \tilde{K} \tag{3.5}$$

Hence we have,

$$\mathfrak{g} = \tilde{K} \oplus \tilde{U}_A \oplus S_{AB} \oplus U_A \oplus K$$

Commutation relations among these generators in terms of the triple product of  $\mathcal{F}$  was given in [7] following earlier references [18, 19].

As was shown in [7] one can realize the Lie algebra  $\mathfrak{g}$  as a quasiconformal Lie algebra over a vector space  $\mathcal{Q}$  whose coordinates  $\mathcal{X}$  are labeled by a pair  $(X, x)$ , where  $X \in \mathcal{F}$  and  $x$  is an extra single variable as follows:

$$\begin{aligned} K(X) &= 0 & U_A(X) &= A & S_{AB}(X) &= (A, B, X) \\ K(x) &= 2a & U_A(x) &= \langle A, X \rangle & S_{AB}(x) &= 2 \langle A, B \rangle x \\ \tilde{U}_A(X) &= \frac{1}{2} (X, A, X) - Ax \\ \tilde{U}_A(x) &= -\frac{1}{6} \langle (X, X, X), A \rangle + \langle X, A \rangle x \\ \tilde{K}(X) &= -\frac{1}{6} (X, X, X) + Xx \\ \tilde{K}(x) &= \frac{1}{6} \langle (X, X, X), X \rangle + 2x^2 \end{aligned} \tag{3.6}$$

The symplectic traceless components of  $S_{AB}$  generate the automorphism group  $H$  of the FTS  $\mathcal{F}$  and the trace part ( $\Delta$ ) is the generator that determines the 5-grading.

One defines a quartic norm over the space  $\mathcal{Q}$  as

$$\mathcal{N}_4(\mathcal{X}) := \mathcal{I}_4(X) - x^2 \tag{3.7}$$

and the "distance" between any two points  $\mathcal{X} = (X, x)$  and  $\mathcal{Y} = (Y, y)$  in  $\mathcal{Q}$  as

$$d(\mathcal{X}, \mathcal{Y}) := \mathcal{N}_4(\delta(\mathcal{X}, \mathcal{Y})) \tag{3.8}$$

where  $\delta(\mathcal{X}, \mathcal{Y})$  is the "symplectic" difference vector of two vectors  $\mathcal{X}$  and  $\mathcal{Y}$  :

$$\delta(\mathcal{X}, \mathcal{Y}) = (X - Y, x - y + \langle X, Y \rangle)$$

The invariance of  $d(\mathcal{X}, \mathcal{Y})$  under the action of the automorphism group of  $\mathcal{F}$  and "translations"  $U_A$  and  $K$  is manifest. The generator  $\Delta$  simply rescales  $d(\mathcal{X}, \mathcal{Y})$ , while under the action of the negative grade generators one finds that  $d(\mathcal{X}, \mathcal{Y})$  gets multiplied by functions linear in  $\mathcal{X}$  and  $\mathcal{Y}$ . Hence the quasiconformal group action preserves light-like separations

$$d(\mathcal{X}, \mathcal{Y}) = 0$$

defined by the quartic norm. This is the reason why the above geometric action of  $G$  was called quasiconformal in [7].

Here we should stress an important point. The construction of a simple Lie algebra  $\mathfrak{g}$  over a FTS  $\mathcal{F}$  extends in a straightforward manner to the complex Lie algebra  $\mathfrak{g}(\mathbb{C})$  by complexifying  $\mathcal{F}$ . Then the above realization of the quasiconformal action of  $G$  extends to a quasiconformal action of  $G(\mathbb{C})$ . One can then obtain quasiconformal realizations of different real forms of  $G$  by appropriate restriction of the complex  $G(\mathbb{C})$ .

#### 4. Minimal unitary representations of non-compact groups from their quasiconformal realizations

In this section we shall review the unified construction of minimal unitary representations of noncompact groups obtained by quantization of their geometric realizations as quasiconformal groups following [11] which generalizes earlier results of [8–10]. Consider the 5-graded decomposition of the Lie algebra  $\mathfrak{g}$  of a noncompact group  $G$

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus (\mathfrak{h} \oplus \Delta) \oplus \mathfrak{g}^{+1} \oplus \mathfrak{g}^{+2} \\ \mathfrak{g} &= E \oplus E^\alpha \oplus (J^a + \Delta) \oplus F^\alpha \oplus F \end{aligned} \quad (4.1)$$

where  $\Delta$  is the generator that determines the 5-grading. Generators  $J^a$  of  $\mathfrak{h}$  satisfy

$$[J^a, J^b] = f^{ab}{}_c J^c \quad (4.2a)$$

where  $a, b, \dots = 1, \dots, D = \dim(\mathfrak{h})$ . Let  $\rho$  denote the symplectic representation by which  $\mathfrak{h}$  acts on  $\mathfrak{g}^{\pm 1}$

$$[J^a, E^\alpha] = (\lambda^a)^\alpha{}_\beta E^\beta \quad [J^a, F^\alpha] = (\lambda^a)^\alpha{}_\beta F^\beta \quad (4.2b)$$

where  $E^\alpha$ ,  $\alpha, \beta, \dots = 1, \dots, N = \dim(\rho)$  are generators that span the subspace  $\mathfrak{g}^{-1}$

$$[E^\alpha, E^\beta] = 2\Omega^{\alpha\beta} E \quad (4.2c)$$

and  $F^\alpha$  are generators that span  $\mathfrak{g}^{+1}$

$$[F^\alpha, F^\beta] = 2\Omega^{\alpha\beta} F \quad (4.2d)$$



$\Omega^{\alpha\beta}$  is the symplectic invariant “metric” of the representation  $\rho$ . The positive (negative) grade generators form a Heisenberg subalgebra since

$$[E^\alpha, E] = 0 \tag{4.2e}$$

with the grade +2 (-2) generator  $F$  ( $E$ ) acting as its central charge. The remaining non-vanishing commutation relations of  $\mathfrak{g}$  are

$$\begin{aligned} F^\alpha &= [E^\alpha, F] & [\Delta, E^\alpha] &= -E^\alpha \\ E^\alpha &= [E, F^\alpha] & [\Delta, F^\alpha] &= F^\alpha \\ [E^\alpha, F^\beta] &= -\Omega^{\alpha\beta} \Delta + \epsilon \lambda_a^{\alpha\beta} J^a & [\Delta, E] &= -2E \\ & & [\Delta, F] &= 2F \end{aligned} \tag{4.2f}$$

where  $\epsilon$  is a constant parameter whose value depends on the Lie algebra  $\mathfrak{g}$ .

In the unified minimal unitary realization of noncompact groups [11], negative grade generators are expressed as bilinears of bosonic oscillators  $\xi^\alpha$  satisfying the canonical commutation relations

$$[\xi^\alpha, \xi^\beta] = \Omega^{\alpha\beta} \tag{4.3}$$

and an extra coordinate  $y$ , corresponding to the singlet in their quasiconformal realization<sup>4</sup>

$$E = \frac{1}{2}y^2 \quad E^\alpha = y \xi^\alpha \quad J^a = -\frac{1}{2}\lambda^a_{\alpha\beta} \xi^\alpha \xi^\beta \tag{4.4}$$

The quadratic Casimir operator of the Lie algebra  $\mathfrak{h}$  is

$$\mathcal{C}_2(\mathfrak{h}) = \eta_{ab} J^a J^b \tag{4.5}$$

where  $\eta_{ab}$  is the Killing metric of the subgroup  $H$ , which is isomorphic to the automorphism group of the underlying FTS  $\mathcal{F}$ . The quadratic Casimir  $\mathcal{C}_2(\mathfrak{h})$  is equal to the quartic invariant of  $H$  in the representation  $\rho$  modulo an additive constant that depends on the normal ordering chosen, namely

$$I_4(\xi^\alpha) = S_{\alpha\beta\gamma\delta} \xi^\alpha \xi^\beta \xi^\gamma \xi^\delta = \mathcal{C}_2(\mathfrak{h}) + \mathfrak{c} \tag{4.6}$$

where  $\mathfrak{c}$  is a constant and

$$S_{\alpha\beta\gamma\delta} := \lambda_{a(\alpha\beta} \lambda_{\gamma\delta)}^a$$

The grade +2 generator  $F$  has the general form

$$F = \frac{1}{2}p^2 + \frac{\kappa (\mathcal{C}_2(\mathfrak{h}) + \mathfrak{c})}{y^2} \tag{4.7}$$

where  $p$  is the momentum conjugate to the singlet coordinate  $y$

$$[y, p] = i \tag{4.8}$$

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<sup>4</sup>Here let us emphasize that we are thereby realizing the Heisenberg algebra  $\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}$  in terms of coordinate and momentum operators  $\xi^\alpha$ , modulo a scale coordinate  $y$  which determines the central charge  $E = \frac{1}{2}y^2$ . This is what we mean by *quantization* of the geometric action of the quasiconformal group.

and  $\kappa$  and  $\mathfrak{C}$  are some constants depending on the Lie algebra  $\mathfrak{g}$ . The grade +1 generators are then given by

$$F^\alpha = [E^\alpha, F] = ip \xi^\alpha + \kappa y^{-1} [\xi^\alpha, \mathcal{C}_2] \quad (4.9)$$

For simple or Abelian  $H$  they take the form [11]

$$F^\alpha = ip \xi^\alpha - \kappa y^{-1} \left[ 2(\lambda^a)^\alpha{}_\beta \xi^\beta J_a + C_\rho \xi^\alpha \right] \quad (4.10)$$

where  $C_\rho$  is the eigenvalue of the second order Casimir of  $H$  in the representation  $\rho$  and one finds

$$[E^\alpha, F^\beta] = -\Delta \Omega^{\alpha\beta} - 6\kappa (\lambda^a)^{\alpha\beta} J_a \quad (4.11)$$

where  $\Delta = -\frac{i}{2}(yp + py)$ .<sup>5</sup>

Using the results of [21] one can give a unified realization of all simple Lie algebras in terms of the underlying FTS's  $\mathcal{F}$  [22]. In the most general case one finds that the commutator of  $E^\alpha$  and  $F^\beta$  has the same form as above, namely [22]

$$[E^\alpha, F^\beta] = -\Delta \Omega^{\alpha\beta} - \epsilon (\lambda^a)^{\alpha\beta} J_a$$

where  $\epsilon$  is a constant and  $(\lambda^a)^{\alpha\beta}$  are the matrices of the Lie algebra of automorphism group  $H$  of the underlying FTS  $\mathcal{F}$ .

For simple  $H$  the quadratic Casimir operator of the Lie algebra  $\mathfrak{g}$  is given by [11]

$$\mathcal{C}_2(\mathfrak{g}) = J^a J_a + \frac{2C_\rho}{N+1} \left( \frac{1}{2} \Delta^2 + EF + FE \right) - \frac{C_\rho}{N+1} \Omega_{\alpha\beta} (E^\alpha F^\beta + F^\beta E^\alpha) \quad (4.12)$$

Furthermore one finds that the quadratic Casimir of  $sl(2)$  and the contribution of the coset generators  $F^\alpha$  and  $E^\beta$  to  $\mathcal{C}_2(\mathfrak{g})$  can all be expressed in terms of the quadratic Casimir  $J^a J_a$  of  $H$ :

$$\begin{aligned} \frac{1}{2} \Delta^2 + EF + FE &= \kappa (J^a J_a + \mathfrak{C}) - \frac{3}{8} \\ \Omega_{\alpha\beta} (E^\alpha F^\beta + F^\beta E^\alpha) &= 8\kappa J^a J_a + \frac{N}{2} + \kappa C_\rho N \end{aligned} \quad (4.13)$$

and the quadratic Casimir of  $\mathfrak{g}$  reduces to a c-number [11]

$$\mathcal{C}_2(\mathfrak{g}) = \mathfrak{C} \left( \frac{8\kappa C_\rho}{N+1} - 1 \right) - \frac{3}{4} \frac{C_\rho}{N+1} - \frac{N}{2} \frac{C_\rho}{N+1} - \frac{\kappa C_\rho^2 N}{N+1} \quad (4.14)$$

as required by irreducibility of the minimal representation. This is a general phenomenon for all minimal unitary realizations of simple groups  $G$  [8–11, 22].

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<sup>5</sup>In this section we follow the conventions of [11]. The indices  $\alpha, \beta, \dots$  are raised and lowered with the antisymmetric symplectic metric  $\Omega^{\alpha\beta} = -\Omega^{\beta\alpha}$  that satisfies  $\Omega^{\alpha\beta} \Omega_{\gamma\beta} = \delta_\gamma^\alpha$  and  $V^\alpha = \Omega^{\alpha\beta} V_\beta$ , and  $V_\alpha = V^\beta \Omega_{\beta\alpha}$ .

## 5. Mapping between Killing potentials in HSS and generators of minimal unitary representations of isometry groups of $\sigma$ -models

To establish a precise mapping between the Killing potentials of the isometry group  $G$  of the sigma model in harmonic superspace and the generators of the minimal unitary realization of  $G$  we shall rewrite the Killing potentials in an  $SU(2)_A$  invariant manner by contracting the generators given in section 4 with the spherical harmonics  $u^{+i}$  and  $u^{-i}$ . First let us define<sup>6</sup>

$$\sqrt{2}q^{+i}u_i^- := w_c \tag{5.1}$$

$$\sqrt{2}q^{+i}u_i^+ := p_c \tag{5.2}$$

The poisson brackets of  $q^{+i}$

$$\{q^{+i}, q^{+j}\} = -\frac{1}{2}\epsilon^{ij} \tag{5.3}$$

imply that

$$\{w_c, p_c\} = -1 \tag{5.4}$$

Under the conjugation  $\widetilde{\phantom{x}}$  we have

$$\widetilde{q^{+i}} = -q^{+i}$$

$$\widetilde{u_i^\pm} = -u_i^\pm$$

which imply

$$\widetilde{w_c} = w_c \tag{5.5}$$

$$\widetilde{p_c} = p_c \tag{5.6}$$

The Hamiltonian can then be written as

$$\mathcal{L}^{+4} = \frac{2P^{+4}(Q^+)}{w_c^2} \tag{5.7}$$

The  $SU(2)_A$  invariant Killing potentials that generate the isometry group  $G$  are then

$$\mathbf{Sp}(2) : S^{++} := K_{ij}^{++} u^{+i} u^{+j} = p_c^2 - \frac{2P^{+4}(Q^+)}{w_c^2}, \tag{5.8}$$

$$S^0 = K_{ij}^{++} (u^{+i} u^{-j} + u^{+j} u^{-i}) = w_c p_c + p_c w_c \tag{5.9}$$

$$S^{--} = K_{ij}^{++} u^{-i} u^{-j} = w_c^2 \tag{5.10}$$

$$\mathbf{H} : K_a^{++} = t_{\alpha\beta} Q^{+\alpha} Q^{+\beta}, \tag{5.11}$$

$$\mathbf{G}/\mathbf{H} \times \mathbf{Sp}(2) : K_\alpha^+ := K_{i\alpha}^{++} u^{+i} = -\sqrt{2} \left\{ p_c Q_\alpha^+ - \frac{1}{w_c} \partial_\alpha^- P^{+4}(Q^+) \right\}, \tag{5.12}$$

$$K_\alpha^- := K_{i\alpha}^{++} u^{-i} = -\sqrt{2} w_c Q_\alpha^+ \tag{5.13}$$

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<sup>6</sup>The  $w_c$  and  $\frac{p_c}{w_c}$  are labelled as fields  $w$  and  $N^{++}$  and interpreted geometrically as the central charge coordinates  $Z^0$  and  $Z^{++}$  in [2].

Comparing the above Killing potentials of the isometry group  $G$  with the generators of the minimal unitary realization of  $G$  given in section 3 we have the following one-to-one correspondence between the elements of harmonic superspace (HSS) and those of minimal unitary realizations (MINREP)

HSS	MINREP
$w_c$	$y$
$p_c$	$p$
$\{, \}$	$i[, ]$
$Q^{+\alpha}$	$\xi^\alpha$
$P^{+4}(Q^+)$	$I_4(\xi)$
$K^{a++} = t_{\alpha\beta}^a Q^{+\alpha} Q^{+\beta}$	$J^a = \lambda_{\alpha\beta}^a \xi^\alpha \xi^\beta$
$K_\alpha^+ = K_{i\alpha}^{++} u^{+i}$	$F^\alpha$
$K_\alpha^- = K_{i\alpha}^{+-} u^{-i}$	$E^\alpha$

The Poisson brackets (PB)  $\{, \}$  in HSS formulation go over to  $i$  times the commutator  $[, ]$  in the minimal unitary realization and the classical harmonic superfields  $w_c, p_c$ , that are canonically conjugate under PB map to the canonically conjugate coordinate and momentum operators  $y, p$ . Similarly, the harmonic superfields  $Q^{+\alpha}$  that form  $n$  conjugate pairs under Poisson brackets go over to the oscillators  $\xi^\alpha$ . This introduces a normal ordering ambiguity in the quantum version of the quartic invariant. Thus the classical expression relating the quartic invariant polynomial  $P^{+4}$  to the quadratic Casimir function in HSS differs from the expression relating the quartic invariant  $I_4$  to the quadratic Casimir of  $H$  by an additive c-number depending on the ordering chosen. The consistent choices for the quadratic Casimir and corresponding c-numbers for all noncompact groups, whose quotients with respect to their maximal compact subgroup are quaternionic symmetric, can be found in [9–11].

The mapping between HSS and MINREP extends also to the equations relating the quadratic Casimir of  $\mathfrak{h}$  to the quadratic Casimir of  $\mathfrak{sp}(2)$  and to the contribution of the coset generators  $G/H \times Sp(2)$  to the quadratic Casimir of  $\mathfrak{g}$  modulo some additive constants due to normal ordering.

Of course on the MINREP side we are working with simple quantum mechanical coordinates and momenta, while in HSS the corresponding quantities are classical harmonic analytic superfields. The easiest way to make more concrete the above mapping is to reduce the  $4d$   $N = 2$   $\sigma$  model to one dimension and quantize it to get a supersymmetric quantum mechanics (with 8 superscharges). What the above mapping implies is that the spectrum of the corresponding quantum mechanics must furnish a minimal unitary representation of the isometry group, which is fully supersymmetric, since the supersymmetry generators commute with the isometry group.

## 6. Discussion

We find the correspondence between the formulation of  $N = 2$ ,  $d = 4$  quaternionic Kähler  $\sigma$  models in HSS and the minimal unitary realizations of their isometry groups established

above quite remarkable. We will discuss some of the implications of this correspondence and open problems, that will be the subjects of separate investigations.

It is important to extend the correspondence between the minimal unitary representation of the isometry group and the classical  $N = 2, d = 4$  quaternionic Kähler  $\sigma$  model to its quantum theory in HSS. There is a subtle issue regarding the quantum implementation of the conjugation  $\tilde{\phantom{x}}$  with respect to which the harmonic derivative  $D^{++}$  is real. This extension to the quantum theory and resolution of the subtle issues should be easier if one reduces the quaternionic Kähler  $N = 2$   $\sigma$  model to two dimensions or to quantum mechanics with eight supersymmetries [23]. For the following discussion we shall assume that there is no obstruction to extending the mapping to the quantum theory.

The correspondence established for symmetric space theories implies that the *fundamental spectra* of the quantum  $N = 2, d = 4$  quaternionic Kähler  $\sigma$  models in  $d = 4$  and their lower dimensional counterparts must fit into the minimal unitary representations of their isometry groups. By the fundamental spectra we mean the well-defined states created by the action of harmonic analytic superfields at a given point in analytic superspace with coordinates  $\zeta^M$  on the vacuum of the theory. From the mapping above we expect that the states created by the purely bosonic components of the analytic superfields will fit into the minimal unitary representation of the corresponding isometry group. Since the analytic superfields are unconstrained, the bosonic spectrum extends to an  $N = 2$  supersymmetric spectrum ( 8 supercharges) by the action of the fermionic components of the superfields.

Now the minimal unitary representations are the analogs of the singleton representations of symplectic groups  $Sp(2n, \mathbb{R})$ . The singleton realizations of  $\mathfrak{sp}(2n, \mathbb{R})$  are free field realizations, i.e. their generators can be written as bilinears of bosonic oscillators. As a consequence the tensoring procedure becomes simple and straightforward for the symplectic groups [24]. However, for other groups the minimal unitary realization is "interacting" and the corresponding generators are nonlinear in terms of the oscillators. This makes the tensoring problem highly nontrivial. The tensoring of Fock spaces of free bosons in the case of  $Sp(2n, \mathbb{R})$  will go over to tensoring of corresponding minimal unitary representations for other noncompact groups. For the quantum  $N = 2$  quaternionic Kähler  $\sigma$  models one then has to tensor the fundamental supersymmetric spectra with each other repeatedly. By an abuse of terminology we shall refer to the resulting spectra as "perturbative" spectra in quantum HSS. The "nonperturbative" spectra in quantum HSS will, in general, contain states that do not form full  $N = 2$  supermultiplets.

The fundamental spectrum is generated by the action of analytic harmonic superfields involving an infinite number of auxiliary fields. Once the auxiliary fields are eliminated the dynamical components of the superfields become complicated nonlinear functions of the physical bosonic and fermionic fields. Therefore the fundamental spectrum in HSS correspond to states created by some complicated nonlinear functions of the physical fields in general. Hence the "fundamental spectrum" is in general not the simple Fock space of free bosons and fermions.

Since the HSS formulation extends to all  $N = 2$  supersymmetric  $\sigma$  models in  $d = 4$ , we expect the fundamental spectra of all  $N = 2, \sigma$  models with nontrivial isometry groups to form minimal unitary representations of their isometry groups. An important class

of  $N = 4$ ,  $\sigma$  models in  $d = 3$ , whose scalar manifolds are not homogeneous, but have interesting isometry groups can be obtained by dimensional reduction (CR-map = C-map times R-map) from unified  $N = 2$  Maxwell-Einstein supergravity theories in  $d = 5$  [5, 25]. These unified  $d = 5$ ,  $N = 2$  MESGTs with simple U-duality groups belong to three infinite families plus a sporadic one. The sporadic theory and the lowest members of the three infinite families are the magical MESGTs whose scalar manifolds are symmetric spaces [26] in 5, 4 and 3 dimensions. The scalar manifolds of the other unified theories in  $d = 5$  are neither symmetric nor homogeneous [25]. The resulting three dimensional  $N = 4$ , quaternionic Kähler  $\sigma$  models can then be lifted up to four dimensional  $N = 2$  supersymmetric  $\sigma$  models, with a rich family of interesting isometry groups.

The  $N = 2$ ,  $d = 4$  MESGT's lead to  $N = 4$ ,  $d = 3$  supersymmetric  $\sigma$  models with quaternionic Kähler manifolds  $\mathcal{M}_3$  under dimensional reduction on a spacelike circle (C-map). On the other hand the stationary black hole solutions of  $N = 2$  MESGTs can be reduced to three Euclidean dimensions on a timelike circle. The resulting theory is  $d = 3$  Euclidean gravity coupled to a para-quaternionic Kähler manifold  $\mathcal{M}_3^*$  [27, 28]. For radially symmetric stationary (supersymmetric) black holes the attractor equations become equivalent to (supersymmetric) geodesic motion on  $\mathcal{M}_3^*$  [29]. The radial quantization of BPS black hole solutions can then be implemented by replacing functions on classical phase space on  $\mathcal{M}_3^*$  by square integrable functions on  $\mathcal{M}_3$  [27, 28]. Furthermore, the  $8n$  dimensional general phase space is reduced to  $4n + 2$  dimensional subspace after imposing the BPS conditions, which can be identified with the twistor space of  $\mathcal{M}_3$  [28, 30]. For very special symmetric quaternionic Kähler manifolds that are obtained from  $d = 5$ ,  $N = 2$  MESGTs by dimensional reduction to three dimensions the corresponding manifolds are of the form

$$\mathcal{M}_3 = \frac{QConf(J)}{Konf(J) \times SU(2)} \tag{6.1}$$

$$\mathcal{M}_3^* = \frac{QConf(J)}{Conf(J) \times Sl(2, \mathbb{R})} \tag{6.2}$$

where  $QConf(J)$  and  $Conf(J)$  denote the quasiconformal and conformal groups of the Jordan algebras  $J$  of degree three that define the corresponding five dimensional theory, respectively.  $Konf(J)$  refers to the compact form of the conformal group  $Conf(J)$  of the Jordan algebra  $J$ .  $Conf(J)$  has been proposed as a spectrum generating symmetry group of the 5d,  $N = 2$  MESGT [31, 7, 32] and  $QConf(J)$  has been proposed as the spectrum generating symmetry group of 4d,  $N = 2$  MESGT defined by  $J$  [7, 32]. The twistor space on  $\mathcal{M}_3$  is simply

$$\frac{QConf(J)}{Konf(J) \times U(1)}$$

whose Kähler potential is given by the distance function that defines the quartic light-cone [33, 28]. Hence the BPS Hilbert space must form a unitary representation of  $G_3$  induced by the geometric realization of  $G_3$  as a quasiconformal group [28]. The unitary representations that arise this way belong to the quaternionic discrete series and are not the minimal unitary representations whose Gelfand-Kirillov dimensions are much smaller. Since the BPS states correspond to four supercharges this result is not surprising. However,

the results presented above imply that the  $N = 4$ ,  $d = 3$  symmetric quaternionic Kähler  $\sigma$  models have *fundamental spectra* which preserve all the supersymmetries. Whether there exist fully supersymmetric black hole solutions belonging to the fundamental spectra in these theories is currently under investigation [28].

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